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Math12 Honors: HW 3.3 Solving Problems and Equations with Polynomial Functions

1. Find the equation of the polynomial with the given information. Then find the relative max/min:

a) zeroes: -1, 2, 2 & Y-intercept: (0,-4)

$y=-(x+1)(x-2)^2$ $a=-1$ since $-1\cdot 1\cdot(-2)^2=4$

$y=-x^3+3x^2-4$

$y'=-3x^2+6x$

$y'=0 \Rightarrow -3x^2+6x=0$

$3x(-x+2)=0$

$x_1=0$ $x_2=2$

$y_1=-4$ $y_2=0$

Relative max: $(2,0)$
Relative min: $(0,-4)$

b) Roots: -3, 2, 5, $f(3)=3$

$y=a(x+3)(x-2)(x-5)$

$f(3)=a(6)(1)(-2)=3 \Rightarrow a=-\frac{1}{4}$

$y=-\frac{1}{4}(x+3)(x-2)(x-5)$

$=-\frac{1}{4}x^3+x^2+\frac{11}{4}x-\frac{15}{2}$

$y'=-\frac{3}{4}x^2+2x+\frac{11}{4}$

$y'=0 \Rightarrow -\frac{3}{4}x^2+2x+\frac{11}{4}=0$

$-3x^2+8x+11=0 \Rightarrow (x+1)(-3x+11)=0 \Rightarrow x_1=-1$, $x_2=\frac{11}{3}$

$y_1=-9$, $y_2=\frac{100}{27}$

Local max: $\left(\frac{11}{3},\frac{100}{27}\right)$
Local min: $(-1,-9)$

c) X-int: -3, 3, 8 Y-int: (0,10)

$y=\frac{5}{36}(x+3)(x-3)(x-8)$

$y=\frac{5}{36}(x^2-9)(x-8)=\frac{5}{36}x^3-\frac{10}{9}x^2-\frac{5}{4}x+10$

$y'=\frac{5}{12}x^2-\frac{20}{9}x-\frac{5}{4}=0$

$15x^2+80x-45=0$

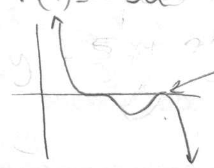
$x=\frac{80\pm\sqrt{80^2+4(15)(45)}}{30}=\frac{80\pm10\sqrt{91}}{30}=\frac{8\pm\sqrt{91}}{3}$

LOCAL min: $\left(\frac{8+\sqrt{91}}{3},-7.53\right)$ LOCAL max: $\left(\frac{8-\sqrt{91}}{3},10.33\right)$

d) Dbl Rt: 4, Triple Rt: 2 $P(1)=5$

$y=a(x-4)^2(x-2)^3$

$P(1)=-9a=5 \Rightarrow a=-\frac{5}{9}$



double root at 0,

Local max: $(4,0)$
Local min: $(3.2,-6.144)$

(using graphing calculator)

e) Zeroes: 0, 0, 1, 1, 1, 2 & $f(-1)=12$

$f(x)=ax^2(x-1)^3(x-2)$

$f(-1)=a(-8)(-3)=12 \Rightarrow a=\frac{1}{2}$

$f(x)=\frac{1}{2}x^2(x-2)(x-1)^3$

local min: $(x,y)=(0,0),(1.795,-0.1659)$

local max: $(x,y)=(0.37,0.028)$

(using graphing calculator)

f) $P(1)=4$, $P(-2)=-11$, $P(3)=34$ Degree of 3.

$a+b+c+d=4$

$-8a+4b-2c+d=-11$

$27a+9b+3c+d=34$

3 equations, 4 variables. Multiple equations possible.

2. If the binomial $x-r$ is a factor of the polynomial $P(x)$, that what is the value of $P(r)$?

$P(r)=0$ Plugging in a root returns zero.

3. Find all the values of "m" which will make $x+2$ a factor of $x^3+3m^2x^2+mx+4$

$f(x)=x^3+3m^2x^2+4$

$f(-2)=-8+12m^2-2m+4=0 \Rightarrow 6m^2-m-2=0 \Rightarrow (2m+1)(3m-2)=0 \Rightarrow m_1=-\frac{1}{2}$ $m_2=\frac{2}{3}$

4. If $x-k$ is a factor of $2x^3-kx^2+(1-k^2)x+5$. What is the value of "k"?

$f(k)=0 \Rightarrow \cancel{2k^3}-\cancel{k^3}+(1-k^2)k+5=0 \Rightarrow k+5=0 \Rightarrow k=-5$

$k-k^3$

5. Solve for the roots of the following polynomial functions:

a) $x^3+3x^2-16x-48=0$

$f(4)=0$ factor: $(x-4)$

4	1	3	-16	-48
	↓	4	28	48
	1	7	12	0

$y=(x-4)(x^2+7x+12)$

$y=(x-4)(x+3)(x+4)$

$\boxed{\text{Roots: } (x,y)=(4,0),(-3,0),(-4,0)}$

b) $6x^3-13x^2+4=0$

$f(2)=0$

2	6	-13	0	4
	↓	12	-2	-4
	6	-1	-2	0

$y=(x-2)(6x^2-x-2)$

$y=(x-2)(2x+1)(3x-2)$

$\boxed{\text{Roots: } (x,y)=(2,0),(-\tfrac{1}{2},0),(\tfrac{2}{3},0)}$

c) $3x^4-5x^2+x+5=0$

$\boxed{\text{NO real roots}}$

d) $2x^3-x^2+3x-6=0$

undfactorable...?

Roots: $(x,y)=(1.24,0)$

no other real roots

e) $x^3+x^2-8x-12=0$

$f(-2)=0$

-2	1	1	-8	-12
	↓	-2	2	12
	1	-1	-6	0

$y=(x+2)(x^2-x-6)$

$y=(x+2)(x-3)(x+2)=(x+2)^2(x-3)$

$\boxed{\text{Roots: } (x,y)=(-2,0),(3,0)}$

f) $6x^3+11x^2-4x-4=0$

$f(-2)=0$

-2	6	11	-4	-4
	↓	-12	2	4
	6	-1	-2	0

$y=(x+2)(6x^2-x-2)$

$y=(x+2)(2x+1)(3x-2)$

$\boxed{\text{Roots: } (x,y)=(-2,0),(-\tfrac{1}{2},0),(\tfrac{2}{3},0)}$

g) $\underbrace{x^3+3x^2+3x+1}_{(x+1)^3}=343$

$(x+1)^3=343$

$x+1=7$

$\boxed{x=6}$

$\boxed{(x,y)=(6,0)}$

Roots

h) $x^5-5x^4+10x^3-10x^2+5x-2=31$

$x^5-5x^4+10x^3-10x^2+5x-1=32$

$(x-1)^5=32$

$x-1=2$

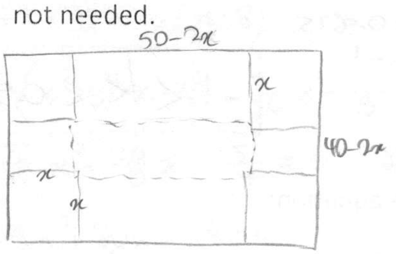
$\boxed{x=3}$

$\boxed{(x,y)=(3,0)}$

6. Find the relative max and min for each of the following cubic functions

a) $f(x)=2x^3-3x^2-8x+12$ $f'(x)=6x^2-6x-8=0$ $3x^2-3x-4=0$ $x=\frac{3\pm\sqrt{9+48}}{6}=\frac{3\pm\sqrt{57}}{6}$ local min: $\left(\frac{3+\sqrt{57}}{6},-0.47\right)$, local max: $\left(\frac{3-\sqrt{57}}{6},15.47\right)$	b) $f(x)=20x^3+17x^2-40x+12$ $f'(x)=60x^2+34x-40$ $30x^2+17x-20=0$ $x=\frac{-17\pm\sqrt{17^2+4(30)(20)}}{60}=\frac{-17\pm\sqrt{2689}}{60}$ local min: $\left(\frac{-17+\sqrt{2689}}{60},-1.579\right)$, local max: $\left(\frac{-17-\sqrt{2689}}{60},50.065\right)$
c) $f(x)=2x^3+x^2-25x+12$ $f'(x)=6x^2+2x-25=0$ $x=\frac{-2\pm\sqrt{4+4(6)(25)}}{12}=\frac{-1\pm\sqrt{151}}{6}$ local min: $\left(\frac{-1+\sqrt{151}}{6},-18.176\right)$, local max: $\left(\frac{-1-\sqrt{151}}{6},50.547\right)$	d) $f(x)=x^3+9x^2+26x+24$ $f'(x)=3x^2+18x+26$ $x=\frac{-18\pm\sqrt{18^2-4(3)(26)}}{6}=\frac{-18\pm\sqrt{12}}{6}=\frac{-9\pm\sqrt{3}}{3}$ local min: $\left(\frac{-9+\sqrt{3}}{3},-0.385\right)$, local max: $\left(\frac{-9-\sqrt{3}}{3},0.385\right)$

7. A piece of card board is 40meters by 50meters. Four squares are cut out from the corners to create a box. What is the largest possible volume of a box that can be created from the piece of cardboard? The top side is not needed.



$V=l\cdot h\cdot w=x(50-2x)(40-2x)$

$=4x(25-x)(20-x)$

$y=4x^3-180x^2+2000x$

$y'=12x^2-360x+2000$

$12x^2-360x+2000=0$

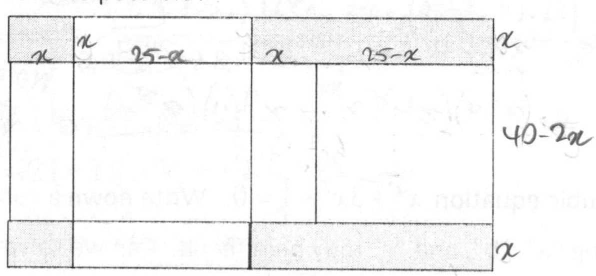
$x=\frac{360\pm\sqrt{360^2-4(12)(2000)}}{24}=\frac{360\pm40\sqrt{21}}{24}$

$=\frac{45\pm5\sqrt{21}}{3}$

local max: $\left(\frac{45-5\sqrt{21}}{3},6564.226\right)$

MAX AREA $=6564.226\ m^3$

8. If the same piece of cardboard is cut like the diagram below to make an addition top side, what is the largest volume possible?



$V=(25-x)(40-2x)\cdot x$

$y=2x^3-90x^2+1000x$

$y'=6x^2-180x+1000=0$

$x=\frac{180\pm\sqrt{180^2-4\cdot6\cdot1000}}{12}=\frac{180\pm20\sqrt{21}}{12}=\frac{45\pm5\sqrt{21}}{3}$

$\boxed{\text{local max: }\left(\frac{45-5\sqrt{21}}{3},3282.113\right)}$

$\boxed{\text{MAX AREA}=3282.113\ m^3}$

9. Given that $x^2-4,\ x^2-2x,\ and\ x^2+x-2$ are all factors of the polynomial $y=f(x)$. What is the lowest possible degree of the polynomial? What is the equation?

$x^2-4=(x-2)(x+2)$

$x^2-2x=x(x-2)$

$x^2+x-2=(x+2)(x-1)$

Unique factors: $(x-2),(x+2),x,(x-1)$

least possible degree of polynomial = 4

Equation: $f(x)=x(x-2)(x+2)(x-1)$

$\boxed{f(x)=x^4-x^3-4x^2+4x}$

10. The polynomial $f(x)$ satisfies the equation $f(x)-f(x-2)=(2x-1)^2$ for all "x". If "p" and "q" are the coefficients of x^2 and x, respectively, in $f(x)$, then what is the value of $p+q$?

$f(x) = ax^3 + px^2 + qx + c$

$f(x) - f(x-2) = (2x-1)^2 \Rightarrow \cancel{ax^3} + \cancel{px^2} + \cancel{qx} + \cancel{c} - \cancel{ax^3} + 6ax^2 - 12ax + 8a - \cancel{px^2} + 4px - 4p - \cancel{qx} + 2q - \cancel{c}$

$6ax^2 + 4x(-3a+p) + 8a - 4p + 2q = 4x^2 - 4x + 1$

$6a = 4 \qquad -3a + p = -1 \qquad 8a - 4p + 2q = 1$

$a = \frac{2}{3} \qquad -3(\frac{2}{3}) + p = -1 \Rightarrow p = 1 \qquad 8(\frac{2}{3}) - 4(1) + 2q = 1$

$q = -\frac{1}{6}$

$p + q = 1 - \frac{1}{6} = \boxed{\frac{5}{6}}$

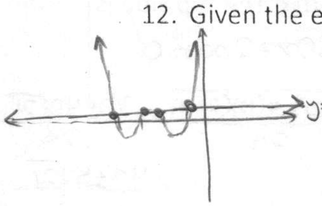
11. Find the range of the function: $h(y) = 2y^4 - 9y^3 + 14y^2 + 6y - 63$

The range of h(y) will be equal to the domain of $h^{-1}(y) = h(x)$.

$h(x) = 2x^4 - 9x^3 + 14x^2 + 6x - 63$

D: $x \in \mathbb{R}$ so $\boxed{\text{Range: } y \in \mathbb{R}}$

12. Given the equation $(x+1)(x+2)(x+3)(x+4) = k$, for what values of "K" will there be four solutions?



$f(x) = (x+1)(x+2)(x+3)(x+4)$ local max = 0.5625

$f(x) = x^4 + 10x^3 + 35x^2 + 50x + 24$ local min = −1

$f'(x) = 4x^3 + 30x^2 + 70x + 50 = 0$

$\boxed{-1 < k < 0.5625}$

$x_1 = -3.618034 \quad x_2 = -2.5 \quad x_3 = -1.381966$

$y_1 = -1 \quad y_2 = 0.5625 \quad y_3 = -1$

13. Challenge: Find (with proof) the product of all the real solutions of the equation:

$(x^{101} - 4x^{99}) + (x^{98} - 4x^{96}) + (x^{95} - 4x^{93}) + \ldots + (x^5 - 4x^3) + (x^2 - 4) = 0$

$x^{99}(x^2-4) + x^{96}(x^2-1) + \cdots + x^3(x^2-4) + (x^2-4) = 0$

roots = 2, −2, −1

$(x^2-4)(x^{99} + x^{96} + x^{93} + \cdots + x^3 + 1) = 0$

$\boxed{\text{Product} = 4}$

$a = 1, \; r = x^3, \; n = 34$

$= (x^2-4)\left(\frac{x^{102}-1}{x^3-1}\right) = \frac{(x^2-4)}{(x^3-1)} \cdot (x^{51}-1)(x^{51}+1) = 0 \Rightarrow \frac{\underbrace{(x^2-4)}_{x=2,-2}(x^3-1)\overbrace{(x^{134} + x^7 + 1)}^{\text{imaginary}}\underbrace{(x^{51}+1)}_{x=-1}}{(x^3-1)} = 0$

NOTE: We can use Vieta here b/c Vieta also counts the complex roots!

14. Challenge: Let "a", "b", and "c" be the roots of the cubic equation $x^3 + 3x^2 - 1 = 0$. Write down a cubic polynomial whose roots are $a^2, b^2, \textit{and } c^2$. Finding "a", "b", and "c" may be difficult. Can we solve the problem without doing that? PIMS UBC

Roots a, b, c

$-3 = a + b + c \Rightarrow a^2 + b^2 + c^2 = (a+b+c)^2 - 2(ab+ac+bc) = 9 - 0 = 9$

$0 = ab + ac + bc \Rightarrow (ab+ac+bc)^2 - 2(a+b+c) \cdot abc = a^2b^2 + a^2c^2 + b^2c^2 = 0^2 - 2(-3)(1) = 6$

$1 = abc \Rightarrow 1 = a^2b^2c^2$

Roots a^2, b^2, c^2

$-\frac{b}{a} = 9 = a^2 + b^2 + c^2$

$+\frac{c}{a} = 6 = a^2b^2 + a^2c^2 + b^2c^2$

$-\frac{d}{a} = 1 = a^2b^2c^2$

$a = 1$

$\boxed{p(x) = x^3 - 9x^2 + 6x - 1}$